On the construction of fuzzy measures for the analysis of poverty and social exclusion

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Summary: This paper is a contribution to the analysis of deprivation seen as a multi-dimensional condition. Multi-dimensionality involves both monetary and diverse non-monetary aspects – the former as the incidence and intensity of low income, and the latter as a lack of access to other resources, facilities, social interactions and even individual attributes determining the life-style. A most useful tool for such analysis is to view deprivation as a matter of degree, giving a quantitative expression to its intensity for individuals in different dimensions and at different times. Such ‘fuzzy’ conceptualisation has been increasingly utilised in poverty and deprivation research. This paper aims to further develop and refine this strand of research, so as to integrate it in the form of a more ‘integrated fuzzy and relative’ (IFR) approach to the analysis of poverty and deprivation. The concern of the paper is primarily methodological rather than detailed numerical analysis from particular applications. We re-examine the two additional aspects introduced by the use of fuzzy (as distinct from the conventional poor/non-poor dichotomous) measures, namely: the choice of membership functions and the choice of rules for the manipulation of the resulting fuzzy sets, rules defining their complement, intersection, union and averaging. The relationship of the proposed fuzzy monetary measure with the Lorenz curve and the Gini coefficient.

Keywords: income poverty, multidimensional deprivation, fuzzy set operators.

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The paper is the results of the common work of all the authors; in particular G. Betti has written Sections 2.3, 3.1 and 4.3; B. Cheli has written Sections 2.1, 3.2 and 4.1; A. Lemmi has written Sections 1, 4.4 and 5; V. Verma has written Sections 2.2, 3.3 and 4.2.
1. Introduction

Most of the methods designed for the analysis of poverty share two main limitations: i) they are unidimensional, i.e. they refer to only one proxy of poverty, namely low income or consumption expenditure; ii) they need to dichotomise the population into the poor and the non-poor by means of the so-called poverty line.

Nowadays many authors recognise that poverty is a complex phenomenon that cannot be reduced to the sole monetary dimension. This leads to the need for a multidimensional approach that consists in extending the analysis to a variety of non-monetary indicators of living conditions.

If multidimensional analysis is increasingly feasible as the available information increases, it was the development of multidimensional approaches that in turn stimulated in many countries the surveying of varied aspects of living conditions.

By contrast, however, little attention has been devoted to the second limitation of the traditional approach, i.e. the rigid poor/non-poor dichotomy with the consequence that most of the literature on poverty measurement continues to be based on the use of poverty thresholds.

Yet it is undisputable that so clear cut a division causes a loss of information and removes the nuances that exist between the two extremes - substantial welfare on the one hand, and distinct material hardship on the other. In other words, poverty should be considered a matter of degree rather than an attribute that is simply present or simply absent for individuals in the population.

An early attempt to incorporate this concept at the methodological level (and in a multidimensional framework) was made by Cerioli and Zani (1990) who drew inspiration from the theory of Fuzzy Sets. Given a set $X$ of elements $x \in X$, any fuzzy subset $A$ of $X$ is defined as follows: $A = \{x, \mu_A(x)\}$, where $\mu_A(x) : X \rightarrow [0,1]$ is called the membership function (m.f.) in the fuzzy subset $A$. The value $\mu_A(x)$ indicates the degree of membership of $x$ in $A$. Thus $\mu_A(x) = 0$ means that $x$ does not belong to $A$, whereas $\mu_A(x) = 1$ means that $x$ belongs to $A$ completely. With $0 < \mu_A(x) < 1$, $x$ belongs to $A$ partially and its degree of membership in $A$ increases in proportion to the proximity of $\mu_A(x)$ to 1.

Cerioli and Zani’s original proposal was later developed by Cheli and Lemmi (1995) giving origin to the so called Totally Fuzzy and Relative (TFR) approach. Both methods have been applied by a number of authors
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subsequently, with a preference for the TFR version\(^1\); in parallel, the TFR method was refined by Cheli (1995) who used it to analyse poverty in fuzzy terms also in the dynamic context represented by two consecutive panel waves.

From this point on, the methodological implementation of this approach has developed in two directions, with somewhat different emphasis despite their common orientation and framework. The first of these is typified by the contributions of Cheli and Betti (1999) and Betti, Cheli and Cambini (2004), focusing more on the time dimension, in particular utilising the tool of transition matrices. The second, with the contributions of Betti and Verma (1999, 2004), has focussed more on capturing the multi-dimensional aspects, developing the concepts of ‘manifest’ and ‘latent’ deprivation to reflect the intersection and union of different dimensions.

In this paper we draw on the state-of-the-art of these developments, to integrate them in the form of an Integrated Fuzzy and Relative (IFR) approach to the analysis of poverty and deprivation. The concern of the paper is primarily methodological but supplemented by numerical analysis from particular applications. We re-examine the two additional aspects introduced by the use of fuzzy (as distinct from the conventional poor/non-poor dichotomous) measures, namely:

(i) the choice of membership functions i.e. quantitative specification of individuals' or households' degrees of poverty and deprivation; and

(ii) the choice of rules for the manipulation of the resulting fuzzy sets, rules defining their complement, intersection, union and averaging.

Moreover, we note the relationship of the proposed fuzzy monetary measure with the Lorenz curve and the Gini coefficient. Certain conceptual and theoretical aspects concerning fuzzy set logic and operations pertinent for the definition of multi-dimensional measures of deprivation are then clarified, and utilised in the construction of a number of such measures. Some of these concepts will be applied to the Italian context on the basis of European Community Household Panel data with reference years 1994-2001.

2. Income poverty

2.1 The conventional income poverty measure (‘Head Count Ratio’)

Diverse ‘conventional’ measures of monetary poverty and inequality are well-known and are not discussed here. In this paper we will focus on only

\(^1\) For instance, Chiappero Martinetti (1994), Quizilbash (2003) and Lelli (2001) use the TFR method in order to analyse poverty or well-being according to Sen’s capability approach.
the most commonly used indicator, namely the proportion of a population
classified as ‘poor’ in purely relative terms on the following lines. To
dichotomise the population into the ‘poor’ and the ‘non-poor’ groups, each
person $i$ is assigned the equivalised income $y_i$ of the person’s household.\(^2\)
Persons with equivalised income below a certain threshold or poverty line
(such as 60% of the median equivalised income) are considered to be poor
(assigned a poverty index, say, $H_i = 1$), and the others as non-poor (assigned
a poverty index $H_i = 0$). The conventional income poverty rate (the Head
Count Ratio, $H$) is the estimated population average of this poverty index,
appropriately weighted by sample weights ($w_i$).

2.2 The propensity to income poverty (‘Fuzzy Monetary’)

Apart from the various methodological choices involved in the construction
of conventional poverty measures, the introduction of fuzzy measures brings
in additional factors on which choices have to be made. These concern at
least two aspects:

Choice of membership functions, meaning a quantitative specification of
the propensity to poverty or deprivation of each person given the level and
distribution of income and other aspects of living conditions of the
population.

Choice of rules for manipulation of the resulting fuzzy sets, specifically
the rules defining complement, intersection, union and aggregation of the
sets.

To be meaningful both these choices must meet some basic logical and
substantive requirements. It is also desirable that they be useful in the sense
of elucidating aspects of the situation not captured (or not captured as
adequately) by the conventional approach.

We begin with the issue of choice of the poverty membership function
(m.f.). In the conventional head count ratio $H$, the m.f. may be seen as
$\mu(y_i) = 1$ if $y_i < z$, $\mu(y_i) = 0$ if $y_i \geq z$, where $y_i$ is equivalised income
of individual $i$, and $z$ is the poverty line. In order to move away from the
poor/non-poor dichotomy, Cerioli and Zani (1990) proposed the introduction
of a transition zone ($z_1 - z_2$) between the two states, a zone over which the

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\(^2\) Equivalised income is defined as the net disposable total household income divided
by equivalised household size, which takes into account variations in household size
and composition. For numerical applications in this paper, we have used the
‘Eurostat’ or ‘modified-OECD’ scale. This scale assigns a weight of 1.0 to the first
adult (aged 14 or more) in the household, 0.5 to each subsequent adult, and 0.3 to
each child in computing the equivalised household size.
m.f. declines from 1 to 0 linearly: \( \mu(y_i) = 1 \) if \( y_i < z_1 \); \( \mu(y_i) = \frac{z_2 - y_i}{z_2 - z_1} \) if \( z_1 \leq y_i < z_2 \); \( \mu(y_i) = 0 \) if \( y_i \geq z_2 \).

In what has been called the ‘Totally Fuzzy and Relative’ approach, Cheli and Lemmi (1995) define the membership function (m.f.) of the set ‘poor’ as the distribution function \( F(y_i) \) of income, normalised (linearly transformed) so as to equal 1 for the poorest and 0 for the richest person in the population. The mean of m.f. so defined is always 0.5.

In order to make this mean equal to some specified value (such as 0.1) so as to facilitate comparison with the conventional poverty rate, Cheli (1995) takes the m.f. as normalised distribution function, raised to some power \( \alpha \geq 1 \):

\[
\mu_i = [1 - F(y_i)]^\alpha = \left( \frac{\sum_j w_j | y_j > y_i}{\sum_j w_j} \right)^\alpha
\]

where persons’ incomes are sorted in increasing order, so that \( y_i \) is the income of the poorest individual.

Increasing the value of this exponent implies giving more weight to the poorer end of the income distribution: empirically, large values of the m.f. would then be concentrated at that end, making the propensity to income poverty sensitive to the location of the poorer persons in the income distribution. Beyond that, the choice of the value of \( \alpha \) is essentially arbitrary, or at best based on some external consideration: this is unavoidable since any method for the quantification of the extent of poverty is inevitably based on the arbitrary choice of some parameter (Hagenaars, 1986). Later Cheli and Betti (1999) and Betti and Verma (1999) chose the parameter \( \alpha \) so that the mean of the m.f. is equal to head count ratio \( H \). This is to facilitate comparison between the conventional and fuzzy measures.

Betti and Verma (1999) define the Fuzzy Monetary indicator (FM), using a somewhat refined version of the above formulation (1):

\[
\mu_i = [1 - L(y_i)]^\alpha = \left( \frac{\sum_j w_j y_i | y_j > y_i}{\sum_j w_j y_i} \right)^\alpha
\]

Here \( L \) represents the Lorenz curve of income. In other terms, \( 1-L(y_i) \) represents the share of the total equivalised income received by all
individuals less poor than the person concerned. It varies from 1 for the poorest, to 0 for the richest individual. \(1-L(y_i)\) can be expected to be a more sensitive indicator of the actual disparities in income, compared to \(1-F(y_i)\), which is simply the proportion of individuals less poor than the person concerned. It may be noted that while the mean of \(1-F(y_i)\) values is \(\frac{1}{2}\) by definition, the mean of \(1-L(y_i)\) values equals \((1+G)/2\), where \(G\) is the Gini coefficient of the distribution.

In this paper we aim to combine the TFR approach of Cheli and Lemmi (1995) and the approach of Betti and Verma (1999) into an 'Integrated Fuzzy and Relative' (IFR) approach. In this approach we take into account both the share of individuals less poor than the person concerned and the share of the total equivalised income received by all individuals less poor than the person concerned. Specifically, the measure is defined as in the following formulation (3):

\[
\mu_i = [1-F(y_i)]^{a-1}[1-L(y_i)] = \frac{\sum_j w_j | y_j > y_i}{\sum_j w_j | y_j > y_1} \left( \frac{\sum_j w_j y_j | y_j > y_i}{\sum_j w_j y_j | y_j > y_1} \right)
\]
Again, parameter $\alpha$ can be chosen so that the mean of the m.f. is equal to head count ratio $H$:

$$E(FM) = \frac{\alpha + G_\alpha}{\alpha(\alpha + 1)} = H$$  \hspace{1cm} (4)

It is elegant that the Fuzzy Monetary (FM) measure as defined above is expressible in terms of the generalised Gini measures. This family of measures (often referred to as ‘s-Gini’) is a generalisation of the standard Gini coefficient, the latter corresponding to $G_1$ with $\alpha=1$. It is defined (in the continuous case) as:

$$G_\alpha = \alpha(\alpha + 1) \int_0^1 [ (1 - F)^{\alpha-1} (F - L) ] dF$$  \hspace{1cm} (5)

The generalised Gini measure weights the distance $F(y_i) - L(y_i)$ between the line of perfect equality and the Lorenz curve by a function of the individual’s position in the income distribution, giving more weight to its poorer end.

### 2.3 Some empirical results

We have calculated the traditional poverty index $H$, and the Fuzzy Monetary index $FM$, for individuals in the Italian European Community Household Panel (ECHP) survey from 1994 to 2001 (income reference years from 1993 to 2000).

Table 1, first panel, shows the conventional poverty rates $H$ for Italy and her five Macro-regions. With the household’s equivalised income ascribed to each of its members, persons with equivalised income below 60% of the national median have been classified as poor. This is done for each ECHP wave separately. The results are also shown averaged over the eight waves so as to gain sampling precision and identify more clearly the overall patterns across Macro-regions.

This suggests the alternative of obtaining a single, more robust value of $\alpha$ by pooling together data from all waves being analysed. This benchmarks the fuzzy poverty rate to be identical to the conventional rate for the group of waves as a whole for all-Italy. This pooled approach has the advantage that only a single parameter has to be estimated, which can therefore be done more reliably.\(^3\) Levels of Macro-regional median incomes are also shown in Table 1 for reference. For the set of Macro-regions considered, there is

\(^3\) Note that in order to define $\alpha$, the quantities $1-L(y_i)$ must still be defined separately for each survey wave. It is only after this, that the data are pooled across waves to determine $\alpha$ iteratively.
generally a *negative* relationship between the income level and the relative poverty rate. This results in part from differences in regional mean incomes (since a common national poverty line has been used), but it also reflects the considerable differences in the levels of inequality within regions. As can be seen from comparing individual cells in the two panels of Table 1, the fuzzy and conventional poverty rates are quite similar to each other. In fact, the ratio \( \frac{F_M}{H} \) is quite stable across waves \( w \) within each Macro-region. The ratio \( \frac{F_M}{H} \) is also similar, though less uniform, across Macro-regions. It also tends to decrease a little with increasing \( H \), meaning that the fuzzy measures show slightly smaller differentials in the Macro-regional poverty rates. Note that the two measures \( F_M \) and \( H \) are constructed, by definition, to be identical at the all-Italy level averaged over the 8 waves.  

### Table 1. Conventional and fuzzy cross-sectional measures of income poverty

<table>
<thead>
<tr>
<th>ECHP wave</th>
<th>w1</th>
<th>w2</th>
<th>w3</th>
<th>w4</th>
<th>w5</th>
<th>w6</th>
<th>w7</th>
<th>w8</th>
<th>8 ECHP waves</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head count ratio (H)</td>
<td>H</td>
<td>EqInc</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Italy</td>
<td>20.4</td>
<td>20.4</td>
<td>20.1</td>
<td>19.7</td>
<td>18.0</td>
<td>18.0</td>
<td>18.5</td>
<td>19.3</td>
<td>19.3</td>
</tr>
<tr>
<td>NORDOVEST</td>
<td>10.2</td>
<td>10.4</td>
<td>9.0</td>
<td>9.2</td>
<td>9.1</td>
<td>6.8</td>
<td>7.0</td>
<td>7.2</td>
<td>9.8</td>
</tr>
<tr>
<td>NORDEST</td>
<td>12.6</td>
<td>10.8</td>
<td>7.8</td>
<td>7.5</td>
<td>6.5</td>
<td>6.6</td>
<td>5.8</td>
<td>5.8</td>
<td>7.9</td>
</tr>
<tr>
<td>CENTRO</td>
<td>14.5</td>
<td>13.2</td>
<td>14.5</td>
<td>15.1</td>
<td>12.8</td>
<td>11.9</td>
<td>13.6</td>
<td>16.1</td>
<td>14.0</td>
</tr>
<tr>
<td>SUD</td>
<td>33.2</td>
<td>34.2</td>
<td>34.2</td>
<td>33.5</td>
<td>30.1</td>
<td>33.0</td>
<td>32.2</td>
<td>33.3</td>
<td>33.0</td>
</tr>
<tr>
<td>ISOLE</td>
<td>40.0</td>
<td>37.2</td>
<td>40.5</td>
<td>37.8</td>
<td>36.3</td>
<td>37.7</td>
<td>40.6</td>
<td>40.8</td>
<td>38.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ECHP wave</th>
<th>w1</th>
<th>w2</th>
<th>w3</th>
<th>w4</th>
<th>w5</th>
<th>w6</th>
<th>w7</th>
<th>w8</th>
<th>8 ECHP waves</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuzzy monetary (FM) poverty rate</td>
<td>FM</td>
<td>FM/H</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Italy</td>
<td>19.4</td>
<td>19.4</td>
<td>19.3</td>
<td>19.3</td>
<td>19.3</td>
<td>19.2</td>
<td>19.2</td>
<td>19.2</td>
<td>19.3</td>
</tr>
<tr>
<td>NORDOVEST</td>
<td>11.5</td>
<td>11.0</td>
<td>10.4</td>
<td>10.3</td>
<td>10.9</td>
<td>9.1</td>
<td>9.1</td>
<td>9.5</td>
<td>10.2</td>
</tr>
<tr>
<td>NORDEST</td>
<td>13.0</td>
<td>11.4</td>
<td>9.9</td>
<td>10.0</td>
<td>8.9</td>
<td>9.2</td>
<td>8.7</td>
<td>8.7</td>
<td>10.0</td>
</tr>
<tr>
<td>CENTRO</td>
<td>15.4</td>
<td>14.7</td>
<td>15.2</td>
<td>16.1</td>
<td>16.4</td>
<td>15.8</td>
<td>16.4</td>
<td>17.2</td>
<td>15.9</td>
</tr>
<tr>
<td>SUD</td>
<td>29.3</td>
<td>30.2</td>
<td>30.2</td>
<td>30.8</td>
<td>30.0</td>
<td>31.8</td>
<td>30.8</td>
<td>30.2</td>
<td>30.4</td>
</tr>
<tr>
<td>ISOLE</td>
<td>34.2</td>
<td>32.9</td>
<td>35.2</td>
<td>32.9</td>
<td>33.9</td>
<td>35.0</td>
<td>36.3</td>
<td>35.9</td>
<td>34.5</td>
</tr>
</tbody>
</table>

HCR = head-count ratio (conventional monetary poverty rate)

FM = fuzzy measure of monetary poverty rate (‘Fuzzy Monetary’)

EqInc = mean equivalised household income (relative to IT mean=100)


### 3. Non-monetary deprivation (“Fuzzy Supplementary”)

#### 3.1 Variables and dimensions of non-monetary deprivation

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This is because this constraint was used in determining the value of the parameter \( \alpha \) as described above.
In addition to the level of monetary income, the standard of living of households and persons can be described by a host of indicators, such as housing conditions, possession of durable goods, the general financial situation, perception of hardship, expectations, norms and values. Quantification and putting together of a large set of non-monetary indicators of living conditions involves a number of steps, models and assumptions.

Firstly, from the large set, which may be available, a selection has to be made of indicators, which are substantively meaningful and useful. For our analysis using the rich ECHP data, a subset of the available indicators was selected. It includes a majority of so-called ‘objective’ indicators of non-monetary deprivation, such as the possession of material goods and facilities and physical conditions of life, at the expense of what may be called ‘subjective’ indicators such as self-assessment of the general health condition, economic hardship and social isolation, or the expressed degree of satisfaction with various aspects of work and life. Although in our opinion these latter types of indicators do provide relevant information on the phenomenon being analysed, several other authors have also considered it preferable to exclude them for various reasons. In particular, it can be empirically observed that the above types of ‘subjective’ indicators tend to be more culture-specific and hence less comparable across countries and regions.

Secondly, it is useful to identify the underlying dimensions and to group the indicators accordingly. Taking into account the manner in which different indicators cluster together (possibly differently in different national situations) adds to the richness of the analysis; ignoring such dimensionality can result in misleading conclusions. In the present analysis we have used the indicators shown in Figure 2, grouped into five dimensions as proposed by Whelan et al. (2001).

### 3.2 Constructing indicators of non-monetary deprivation

Individual items indicating non-monetary deprivation often take the form of simple ‘yes/no’ dichotomies (such as the presence or absence of enforced lack of certain goods or facilities). However, some items may involve more than two ordered categories, reflecting different degrees of deprivation. Consider the general case of \( c = 1 \) to \( C \) ordered categories of some deprivation indicator, with \( c = 1 \) representing the most deprived and \( c = C \) the least deprived situation. Let \( c_i \) be the category to which individual \( i \) belongs. Cerioli and Zani (1990), assuming that the rank of the categories represents an equally-spaced metric variable, assigned to the individual a deprivation score as:

\[
d_i = \frac{(C - c_i)}{(C - 1)}, \quad 1 \leq c_i \leq C
\]
### Dimensions and items of non-monetary deprivation

<table>
<thead>
<tr>
<th>Category</th>
<th>Item Details</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Basic non-monetary deprivation - these concern the lack of ability to afford most basic requirements:</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Keeping the home (household’s principal accommodation) adequately warm.</td>
<td>0.297</td>
</tr>
<tr>
<td></td>
<td>Paying for a week’s annual holiday away from home.</td>
<td>0.193</td>
</tr>
<tr>
<td></td>
<td>Replacing any worn-out furniture.</td>
<td>0.150</td>
</tr>
<tr>
<td></td>
<td>Buying new, rather than second hand clothes.</td>
<td>0.433</td>
</tr>
<tr>
<td></td>
<td>Eating meat chicken or fish every second day, if the household wanted to.</td>
<td>0.759</td>
</tr>
<tr>
<td></td>
<td>Having friends or family for a drink or meal at least once a month.</td>
<td>0.357</td>
</tr>
<tr>
<td></td>
<td>Inability to meet payment of scheduled mortgage payments, utility bills or hire purchase instalments.</td>
<td>1.638</td>
</tr>
<tr>
<td>2</td>
<td>Secondary non-monetary deprivation – these concern enforced lack of widely desired possessions (&quot;enforced&quot; means that the lack of possession is because of lack of resources):</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A car or van.</td>
<td>2.838</td>
</tr>
<tr>
<td></td>
<td>A colour TV.</td>
<td>0.610</td>
</tr>
<tr>
<td></td>
<td>A video recorder.</td>
<td>0.700</td>
</tr>
<tr>
<td></td>
<td>A microwave.</td>
<td>0.518</td>
</tr>
<tr>
<td></td>
<td>A dishwasher.</td>
<td>1.479</td>
</tr>
<tr>
<td></td>
<td>A telephone.</td>
<td>2.472</td>
</tr>
<tr>
<td>3</td>
<td>Housing facilities – these concern the absence of basic housing facilities (so basic that one can presume all households would wish to have them):</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A bath or shower.</td>
<td>3.231</td>
</tr>
<tr>
<td></td>
<td>An indoor flushing toilet.</td>
<td>2.063</td>
</tr>
<tr>
<td></td>
<td>Hot running water.</td>
<td>0.389</td>
</tr>
<tr>
<td>4</td>
<td>Housing deterioration – these concern serious problems with accommodation:</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Leaky roof.</td>
<td>0.396</td>
</tr>
<tr>
<td></td>
<td>Damp walls, floors, foundation etc.</td>
<td>0.655</td>
</tr>
<tr>
<td></td>
<td>Rot in window frames or floors.</td>
<td>0.905</td>
</tr>
<tr>
<td>5</td>
<td>Environmental problems – these concern problems with the neighbourhood and the environment:</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Shortage of space.</td>
<td>0.815</td>
</tr>
<tr>
<td></td>
<td>Noise from neighbours or outside.</td>
<td>0.880</td>
</tr>
<tr>
<td></td>
<td>Dwelling too dark/not enough light.</td>
<td>0.594</td>
</tr>
<tr>
<td></td>
<td>Pollution, grime or other environmental problems caused by traffic or industry.</td>
<td>0.666</td>
</tr>
<tr>
<td></td>
<td>Vandalism or crime in the area.</td>
<td>0.961</td>
</tr>
</tbody>
</table>

**Figure 2.** Dimensions and items of non-monetary deprivation analysed

Cheli and Lemmi (1995) proposed an improvement by replacing the simple ranking of the categories with their *distribution function* in the population:

$$d_i = \frac{1 - F(c_i)}{[1 - F(1)]} \quad (7)$$

Note that the above two formulations for $d_i$ are identical in by far the most common case – that of a dichotomous indicator ($C = 2$), giving a dichotomous m.f. $d_i = 1$ (deprived) or $d_i = 0$ (non-deprived).

The procedure for aggregating over a group of item is also the same for the two formulations: a weighted sum is taken over items ($k$):

$$\mu_i = \frac{\Sigma w_k d_{k,i}}{\Sigma w_k} \quad \text{where the } w_k \text{ are item-specific weights, taken in the above references as } w_k = \ln(1/\bar{d}_k). \quad \text{For dichotomous indicators, } \bar{d}_k \text{, the mean of individual values } (d_i) \text{ for item } k \text{, simply equals the proportion deprived on that item. In our approach here, we use the above framework, but with some important refinements proposed by Betti and Verma (1999,}$$
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2004), which amounts to constructing non-monetary indicators in exactly the same way as the income indicator described in Section 2.

(1) We begin by selecting the items to be included in the index or indices of deprivation on substantive grounds, and grouping the items into ‘dimensions’ as described in Section 3.1. Deprivation scores \( d_{k,i} \) are assigned to ordinal categories of each item as in (6).

(2) The weights to be given to items are determined within each dimension separately as described in Betti and Verma (1999). With these weights, a deprivation score is determined for each dimension \( \delta \): \[ S_{\delta,j} = \frac{\sum w_k (1 - d_{k,j})}{\sum w_k} \] and also for the overall situation covering all the indicators: \[ S_i = \frac{\sum w_k (1 - d_{k,j})}{\sum w_k} \]

Note that \( S \) is a ‘positive’ score indicating lack of deprivation; thus it is akin to income in Section 2.

(3) As in the Fuzzy Monetary approach, we may consider three alternative definitions for the individual’s degree of non-monetary deprivation \( FS_i \). All of them are consistent with a relative concept of deprivation and are analogous to the three shapes specified for monetary deprivation.

Restricting our attention to the set of all indicators, for any individual \( i \), these definitions are:

i) The proportion of individuals who are less deprived than \( i \): \[ \mu_i = FS_i = \left[ 1 - F(S_i) \right]^a \], where \( F(S_i) \) represents the distribution function of \( S \) evaluated for individual \( i \):

\[
\mu_i = \left[ 1 - F(S_i) \right]^a = \left( \frac{\sum w_j \left| S_j > S_i \right|^a}{\sum w_j \left| S_j > S_i \right|} \right)
\]  

ii) The share of the total non-deprivation \( S \) assigned to all individuals less deprived than \( i \): \[ \mu_i = FS_i = \left[ 1 - L(S_i) \right]^a \], where \( L(S_i) \) represents the value of the Lorenz curve of non-deprivation score \( S \) for individual \( i \), calculated according to the form below:  

\[ \mu_i = \left[ 1 - L(S_i) \right]^a \]

---

5 Forms (i) and (ii) have been chosen so as to take into account tied rankings, which are much more frequent for items with a few categories, compared to the case of
Both \( i \) and \( ii \) vary from 0 for the least deprived individual to 1 for the most deprived one, and both of them can be raised to an exponent \( \alpha \geq 1 \) that represents the weight of the poorer people compared to the less poor ones.

\[ \mu_i = [1 - L(S_i)]^\alpha = \left( \frac{\sum w_j S_j \mid S_j > S_i}{\sum w_j S_j \mid S_j > S_i} \right)^\alpha \]  

(11)

Both \( i \) and \( ii \) vary from 0 for the least deprived individual to 1 for the most deprived one, and both of them can be raised to an exponent \( \alpha > 1 \) that represents the weight of the poorer people compared to the less poor ones.

\[ \mu_i = FS_i = [1 - F(S_i)]^\alpha [1 - L(S_i)]; \; \alpha \geq 1 \]  

(12)

It is very likely that in practice, the three formulations lead to very similar results. Nevertheless, on the basis of the same theoretical considerations as noted in the case of income, we prefer to opt for specification \( iii \) which is one of the new proposals of this paper.

As before, we have estimated a single value of parameter \( \alpha \) on the basis of data pooled over waves so as to match the overall \( FS \) rate (i.e., for all-Italy averaged over 8 waves), with the overall averaged monetary poverty rate. This is the same constraint as for \( FM \) in Section 2; of course, the values of the parameter \( \alpha \) to meet this constraint are different in the two cases.

Since the aim of this paper is essentially methodological, in the empirical application below we do not present separate measures of non-monetary deprivation for each dimension \( \delta \) (which would be important on the substantive level), and consider only the index of overall deprivation incorporating all the 24 indicators listed in Section 3.1 above.

### 3.3 Some empirical results

Using the last described approach, Table 2 compares fuzzy measures of income poverty and of non-monetary deprivation across Italian Macro-regions. For reasons noted, the two measures have been averaged over 8 waves, and are scaled to be identical to each other at all-Italy level. Macro-regions with low levels of monetary poverty indicate a higher level of non-monetary deprivation compared to their level of monetary poverty. Overall, there is a notable negative correlation between the level of income poverty (\( FM \)) and the ratio (\( FS/FM \)), though in actual values the two measures (\( FM, FS \)) are quite similar and equally relative.

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continuous variables like income. In fact in practice, we use (1) and (2) for fuzzy monetary in exactly the same form as above.
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We will return to the rest of the table in Section 4.4.

Table 2. Fuzzy measures of deprivation: monetary, non-monetary, and the two forms in combination

<table>
<thead>
<tr>
<th>Fuzzy deprivation rates</th>
<th>Ratios</th>
<th>MF</th>
<th>Manifest</th>
<th>Latent</th>
<th>Mean</th>
<th>MF</th>
<th>Manifest</th>
<th>Latent</th>
<th>Mean</th>
<th>Latent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Italy</td>
<td>19.3</td>
<td>19.3</td>
<td>9.3</td>
<td>29.3</td>
<td>19.3</td>
<td>1.00</td>
<td>0.48</td>
<td>1.52</td>
<td>0.32</td>
<td></td>
</tr>
<tr>
<td>NORDOVEST</td>
<td>10.2</td>
<td>15.0</td>
<td>4.0</td>
<td>21.2</td>
<td>12.6</td>
<td>1.47</td>
<td>0.32</td>
<td>1.68</td>
<td>0.19</td>
<td></td>
</tr>
<tr>
<td>NORDEST</td>
<td>10.0</td>
<td>11.1</td>
<td>3.6</td>
<td>17.5</td>
<td>10.5</td>
<td>1.11</td>
<td>0.34</td>
<td>1.66</td>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td>CENTRO</td>
<td>15.9</td>
<td>16.5</td>
<td>6.7</td>
<td>25.7</td>
<td>16.2</td>
<td>1.04</td>
<td>0.41</td>
<td>1.59</td>
<td>0.26</td>
<td></td>
</tr>
<tr>
<td>SUD</td>
<td>30.4</td>
<td>27.2</td>
<td>16.4</td>
<td>41.2</td>
<td>28.8</td>
<td>0.90</td>
<td>0.57</td>
<td>1.43</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>ISOLE</td>
<td>34.5</td>
<td>28.5</td>
<td>18.4</td>
<td>44.6</td>
<td>31.5</td>
<td>0.82</td>
<td>0.58</td>
<td>1.42</td>
<td>0.41</td>
<td></td>
</tr>
</tbody>
</table>

FM fuzzy measure of monetary poverty rate ('fuzzy monetary')
FS fuzzy measure of overall non-monetary deprivation rate ('fuzzy supplementary')
Manifest propensity to both FM and FS deprivation
Latent propensity to either form of deprivation (FM and/or FS)
Mean mean of (FM,FS) = mean of (Manifest, Latent)

Note: Figures show simple averages of cross-sectional results over 8 ECHP waves

4. Fuzzy set operations appropriate for the analysis of poverty and deprivation

4.1 Multidimensional measures

In the previous sections we have considered poverty as a fuzzy state and defined measures of its degree in monetary and different non-monetary dimensions. In multidimensional analysis it is of interest to know the extent to which deprivation in different dimensions tends to overlap for individuals. Such analyses require the specification of rules for the manipulation of fuzzy sets, such as defining set complements, intersections, unions and aggregations.

As a concrete example let us consider deprivation in two dimensions: income deprivation and overall non-monetary deprivation that we denote by m and s respectively, each of them being characterised by two opposite fuzzy states, labelled as 0 (non-deprivation) and 1 (deprivation), which correspond to a pair of fuzzy sets forming a fuzzy partition.

In terms of the quantities introduced earlier, Table 3 shows the degree of membership of individual i in the sets defining the two dimensions of deprivation and their complements.

---

6 Similarly, in longitudinal analysis it would be of interest to know the extent to which the state of poverty or deprivation persists over time for the person concerned.
In a joint analysis of monetary and non-monetary deprivation, any individual belongs to each of the four sets (representing the intersections \(m \cap s\); \(m = 0,1\); \(s = 0,1\)) with the degree of membership varying between 0 and 1, as reported in Table 4. The degree of membership in \(m \cap s\) of individual \(i\), denoted by \(\mu_{ims}\), represents a measure of the extent to which the individual is affected by the particular combination of states \((m,s)\). The marginal constraints pointed out in Table 4 must be satisfied. Moreover, since these four sets form a fuzzy partition, their respective degrees of membership must sum to 1 for any \(i\).

**Table 3. Membership functions of an individual in the 4 intersection sets**

<table>
<thead>
<tr>
<th>Poverty dimension</th>
<th>State</th>
<th>Individual degree of membership</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m)</td>
<td>non-deprivation 0</td>
<td>(1 - FM_i)</td>
</tr>
<tr>
<td></td>
<td>deprivation 1</td>
<td>(FM_i)</td>
</tr>
<tr>
<td>(s)</td>
<td>non-deprivation 0</td>
<td>(1 - FS_i)</td>
</tr>
<tr>
<td></td>
<td>deprivation 1</td>
<td>(FS_i)</td>
</tr>
</tbody>
</table>

**Table 4. Situation of a generic individual \(i\) seen in fuzzy terms**

<table>
<thead>
<tr>
<th>Monetary deprivation ((m))</th>
<th>Non-monetary deprivation ((s))</th>
<th>poverty status</th>
<th>non-poor (0)</th>
<th>poor (1)</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>non-poor (0)</td>
<td>(\mu_{00})</td>
<td>(\mu_{i01})</td>
<td>1 - (FM_i)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>poor (1)</td>
<td>(\mu_{i10})</td>
<td>(\mu_{i11})</td>
<td>(FM_i)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>1 - (FS_i)</td>
<td>(FS_i)</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Our main goal is to find a specification of \(\mu_{ims}\) that is the most appropriate to our purpose of analysing poverty and deprivation. However, before answering this specific question, let us return to the general issue of defining fuzzy sets operations.

Fuzzy set operations are a generalisation of the corresponding 'crisp' set operations in the sense that the former reduce to (exactly reproduce) the latter when the fuzzy membership functions, being in the whole range \([0,1]\), are reduced to a \(\{0,1\}\) dichotomy. There are, however, more than one ways in which the fuzzy set operations can be formulated, each representing an equally valid generalisation of the corresponding crisp set operations. The choice among alternative formulations has to be made primarily on substantive grounds: some options are more appropriate (meaningful, convenient) than others, depending on the context and objectives of the
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application. While the rules of fuzzy set operations cannot be discussed fully in this paper, we need to clarify their application specifically for the study of poverty and deprivation.

4.2 Intersection, union and complement of fuzzy sets

Since fuzzy sets are completely specified by their membership functions, any operation with them (such as union, intersection and complement) is defined in terms of the membership functions of the original fuzzy sets involved. As an example, membership $\mu_{i11}$ of Table 4 is function of $FM_i$ and $FS_i$, and might be more precisely written as $\mu_{i11} (FM_i, FS_i)$. However in the following discussion it will be convenient to use the following simplified notation: $(a,b)$ for $(FM_i, FS_i)$, the membership functions of two sets for individual $i$ (subscript $i$ can be dropped when not essential); also $s_1=\min(a,b)$ and $s_2=\max(a,b)$. We also denote by $c(\ldots)$, $i(\ldots)$ and $u(\ldots)$ the basic set operations of complementation, intersection and union, respectively. Sometimes the more compact notation $\vec{a}$ will be used in the place of $c(a)$.

Table 5 displays some of the most important ways to specify fuzzy intersection and union that satisfy a set of essential requirements such as ‘reduction to the crisp set operation’, ‘boundary condition’, ‘monotonicity’, ‘cumutativity’, etc. Among them, the so called Standard fuzzy intersection and union are the only ones which satisfy the intuitively and substantively desiderable condition of ‘idempotency’, namely: $i(a,a) = a$ and $u(a,a) = a$.

Besides standard intersection and union, the Standard fuzzy complement is defined as:

$c(a) = 1-a$ and $c(b) = 1-b$,

that is, the same way in which we have defined the membership in the non-poor ($1-FM_i$ and $1-FS_i$) in Section 4.1.

Table 5. Basic forms of fuzzy set intersections and unions

<table>
<thead>
<tr>
<th></th>
<th>intersection</th>
<th>union</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard</td>
<td>$i(a,b) = \min(a,b)$</td>
<td>$u(a,b) = \max(a,b)$</td>
</tr>
<tr>
<td></td>
<td>$= s_1 = i_{\max}$</td>
<td>$= s_2 = u_{\min}$</td>
</tr>
<tr>
<td>Algebraic</td>
<td>$i(a,b) = a*b$</td>
<td>$u(a,b) = a + b - a*b$</td>
</tr>
<tr>
<td>Bounded</td>
<td>$i(a,b) = \max(0, a + b - 1)$</td>
<td>$u(a,b) = \min(1, a + b)$</td>
</tr>
</tbody>
</table>

It must be underlined that permissible forms of the two operations, intersection and union, go in pairs: to be consistent, it is necessary to select the two from the same row of Table 5, so as to satisfy the De Morgan laws.

---

7 For details see Klir and Yuan (1995).
of set operations: 

\[(A \cup B)' = A' \cap B', \quad (A \cap B)' = A' \cup B',\]

which in the fuzzy case can be written as:

\[
c[i(a,b)] = u[c(a), c(b)]; \quad c[u(a,b)] = i[c(a), c(b)].
\]

### 4.3 Definition of poverty measures according to both monetary and non-monetary dimensions

For our application, a most important observation is that the Standard fuzzy operations provide the largest (the most loose or the weakest) intersection and by contrast the smallest (the most tight or the strongest) union among all the permitted forms. It is for this reason that they have been labelled as \(i_{\text{max}}\) and \(u_{\text{min}}\) in Table 5. It is this factor which makes it inappropriate to use the Standard set operations uniformly throughout in our application to poverty analysis. In fact if the Standard operation were applied to all the four intersections of Table 4, their sum would exceed 1 and the marginal constraints would not be satisfied.\(^8\)

Now it can be easily proved that the Algebraic form, applied to all the four intersections, is the only one, which meets this condition. But despite this numerical consistency, we do not regard the Algebraic form to give results, which, for our particular application, would be generally acceptable on intuitive or substantive grounds. In fact, if we take the liberty of viewing the fuzzy propensities as probabilities, then the Algebraic product rule \(i(a,b) = a \ast b\) implies zero correlation between the two forms of deprivation, which is clearly at variance with the high positive correlation we expect in the real situation for similar states. The rule therefore seems to provide an unrealistically low estimate for the resulting membership function for the intersection of two similar states. The Standard rules, giving higher overlaps (intersections) are more realistic for \((a,b)\) representing similar states.

By contrast, in relation to dissimilar states \((\bar{a}, b)\) and \((a, \bar{b})\) (lack of an overlap between deprivations in two dimensions), it appears that the Algebraic rule (and hence also the Standard rules) tend to give unrealistically high estimates for the resulting membership function for the union. The reasoning similar to the above applies: in real situations, we expect large negative correlations (hence reduced intersections) between dissimilar states in the two dimensions of deprivation. In fact, it can be seen, by considering some particular numerical values for \((\bar{a}, b)\) or \((a, \bar{b})\), that Bounded rule, for instance, gives more realistic results for dissimilar states.

Given the preceding considerations, the specification of the fuzzy intersection \(i(a,b)\) that appears to be the most reasonable for our particular

\(^8\) For details, see Betti and Verma (2004).
application and that satisfies the above mentioned marginal constraints is of a 'composite' type as follows (Betti and Verma, 2004):

For sets representing similar states - such as the presence (or absence) of both types of deprivation - the Standard operations (which provide larger intersections than Algebraic operations) are used.

For sets representing dissimilar states - such as the presence of one type but the absence of the other type of deprivation - we use the Bounded operations (which provide smaller intersections than Algebraic operations).

By applying this composite intersection the elements of Table 4 are specified as shown in Table 6. Figure 3 illustrates the Composite operation graphically.

![Composite fuzzy set operations](image)

*Figure 3. The Composite fuzzy set operations*

Note that the propensity to the deprived in at least one of the two dimensions equals $\max(FM_i, FS_i)$, which can be viewed as any of the three entirely equivalent forms:

- as the complement of cell “0-0” in Table 6, or
- as the sum of the membership functions in the other three cells, or
- as the union of $(FM_i, FS_i)$. 
### Table 6. Joint measures of deprivation according to the Betti-Verma Composite operation

<table>
<thead>
<tr>
<th>Monetary deprivation (m)</th>
<th>Non-monetary deprivation (s)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>poverty status</td>
<td>non-poor (0)</td>
</tr>
<tr>
<td>non-poor (0)</td>
<td>min(1−FM, 1−FS)</td>
<td>max(0, FS−FM)</td>
</tr>
<tr>
<td></td>
<td>=1-max(FM, FS)</td>
<td></td>
</tr>
<tr>
<td>poor (1)</td>
<td>max(0, FM−FS)</td>
<td>min(FM, FS)</td>
</tr>
<tr>
<td></td>
<td>total</td>
<td>1 − FS</td>
</tr>
</tbody>
</table>

#### 4.4 Income poverty and non-monetary deprivation in combination: Manifest and Latent deprivation

The two measures – $FM$ the propensity to income poverty, and $FS$ the overall non-monetary deprivation propensity - may be combined to construct composite measures which indicate the extent to which the two aspects of income poverty and non-monetary deprivation overlap for the individual concerned. These measures are as follows.

- **$M_i$**: Manifest deprivation,
  
  representing the propensity to both income poverty and non-monetary deprivation simultaneously. One may think of this as the ‘more intense’ degree of deprivation.

- **$L_i$**: Latent deprivation,

  representing the individual being subject to at least one of the two, income poverty and/or non-monetary deprivation; one may think of this as the ‘less intense’ degree of deprivation.

Once the propensities to income poverty ($FM_i$) and non-monetary deprivation ($FS_i$) have been defined at the individual level $i$, the corresponding combined measures are obtained in a straightforward way, using the Composite set operations defined in the previous section. These can then be aggregated to produce the relevant averages or rates for the population. The Manifest deprivation propensity of individual $i$ is the intersection (the smaller) of the two measures $FM_i$ and $FS_i$:

$$M_i = \min(FM_i, FS_i)$$  \hspace{1cm} (13)

Similarly, the Latent deprivation propensity of individual $i$ is the union (the larger) of the two measures $FM_i$ and $FS_i$:

$$L_i = \max(FM_i, FS_i)$$  \hspace{1cm} (14)
As we have seen above, the estimates provided by the Standard operations are maximal for the intersections and minimal for the union, so that we have a maximal estimate for Manifest deprivation, and a minimal for Latent deprivation. As noted in Section 4.3, we argue that on substantive grounds, this is a reasonable (indeed desirable) choice for intersections of ‘similar’ states.

Returning to Table 2 presented in Section 3.4, the last column of the table shows Manifest deprivation index as percentage of Latent deprivation index: it can be interpreted as an index of the degree of overlap (or intersection), at the level of individual persons, between income poverty and non-monetary deprivation.

In theory, this ratio varies from 0 to 1. When there is no overlap (i.e., when the subpopulation subject to income poverty is entirely different from the subpopulation subject to non-monetary deprivation), Manifest deprivation rate and hence the above mentioned ratio equals 0. When there is complete overlap (i.e., when exactly the same subpopulation is subject both to income poverty and to non-monetary deprivation), the Manifest and Latent deprivation rates are the same and hence the above mentioned ratio equals 1.

It is important to highlight that there is a higher degree of overlap between income poverty and non-monetary deprivation at the level of individual persons in poorer Macro-regions, and a lower degree of overlap in richer Macro-regions of Italy. This leads to the conclusion that the adoption of a multi-dimensional approach is particularly important when analysing richer regions (or for that matter, countries in an international study), where different dimensions have less overlap. Therefore in this case the adoption of a supplementary indicator as a complement to the monetary one is justified, because it has an added value. On the other hand, because of the higher degree of overlap in poorer (and less equal) regions, the overall deprivation is more intense for the subpopulations involved, which is also important. All this underlines the need to supplement monetary indicators by multi-dimensional measures.

5. Concluding remarks

The aim of this paper has been to develop and refine the strand of research, which started with the contributions of Cerioli and Zani (1990) and then has been followed by a number of applications and developments. Methodologically, the implementation of this approach has developed in two directions, with somewhat different emphasis despite their common orientation and framework: one emphasising more the multidimensionality and the other more the longitudinal aspects of poverty analysis. We have tried to bring together various strands of development on the subject.
Specifically, we address in this paper the additional factors, which the introduction of fuzzy approach, as distinct from the conventional approach, brings into the analysis of poverty and deprivation. Choices have to be made in relation to these, involving at least two aspects:

Choice of ‘membership functions’, that is, quantitative specification of the propensity to poverty and deprivation of each person in the population, given the level and distribution of income and other measures of living standards.

Choice of ‘rules’ for manipulation of the resulting fuzzy sets, specifically the rules defining complements, intersections, union and aggregation of the sets.

A major objective has been to clarify how both these choices must meet some basic logical and substantive requirements to be meaningful. We believe that, hitherto, the rules of fuzzy set operations in the context of poverty analysis have not been well or widely understood.

The focus in this paper has been on the multidimensional aspects of deprivation. Persistence and movement over time is an equally important aspect of the intensity of deprivation, requiring longitudinal study at the micro level and in the aggregate. Procedures similar to the ones developed here can also be extended to fuzzy set based longitudinal analysis. We recently developed these aspects further in a separate paper (Betti, Cheli and Verma, 2006).

References


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